AMC 10 - Mathcounts - UKMT

Homework Problems- Hints

February, 2021

Affine Transformations

Problem 1

Let C be the graph of $xy = 1$, and denote by C^* the reflection of C in the line $y = 2x$. Let the equation of C^* be written in the form

$$12x^2 + bxy + cy^2 + d = 0.$$

Find the product bc.
(A) 82 **(B)** 83 **(C)** 84 **(D)** 85 **(E)** 86

- iven a point $P(x, y)$ on C, we look to find a formula for $P'(x', y')$ on C^*. Both points lie on a line that is perpendicular to $y = 2x$, so the slope of $\overline{PP'}$ is $\frac{-1}{2}$. Thus $\frac{y'-y}{x'-x} = \frac{-1}{2} \implies x' + 2y' = x + 2y$. Also, the midpoint of $\overline{PP'}$, $\left(\frac{x+x'}{2}, \frac{y+y'}{2}\right)$, lies on the line $y = 2x$. Therefore $\frac{y+y'}{2} = x + x' \implies 2x' - y' = y - 2x$.
- Solving these two equations, we find $x = \frac{-3x'+4y'}{5}$ and $y = \frac{4x'+3y'}{5}$. Substituting these points into the equation of C, we get $\frac{(-3x'+4y')(4x'+3y')}{25} = 1$, which when expanded becomes $12x'^2 - 7x'y' - 12y'^2 + 25 = 0$.
Thus, $bc = (-7)(-12) = \boxed{084}$.

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Problem 2

A particle is located on the coordinate plane at $(5, 0)$. Define a move for the particle as a counterclockwise rotation of $\frac{\pi}{4}$ radians about the origin followed by a translation of 10 units in the positive x-direction. Given that the particle's position after 150 moves is (p, q), find the greatest integer less than or equal to $|p| + |q|$.

(A) 13 **(B)** 17 **(C)** 19 **(D)** 23 **(E)** 29

- Let $P(x, y)$ be the position of the particle on the xy-plane, r be the length OP where O is the origin, and θ be the inclination of OP to the x-axis. If (x', y') is the position of the particle after a move from P, then we have two equations for x' and y':

$$x' = r\cos(\frac{\pi}{4} + \theta) + 10 = \frac{\sqrt{2}(x - y)}{2} + 10$$

$$y' = r\sin(\frac{\pi}{4} + \theta) = \frac{\sqrt{2}(x + y)}{2}$$

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- Let (x_n, y_n) be the position of the particle after the nth move, where $x_0 = 5$ and $y_0 = 0$. Then $x_{n+1} + y_{n+1} = \sqrt{2}x_n + 10$, $x_{n+1} - y_{n+1} = -\sqrt{2}y_n + 10$. This implies $x_{n+2} = -y_n + 5\sqrt{2} + 10$, $y_{n+2} = x_n + 5\sqrt{2}$.
- Substituting $x_0 = 5$ and $y_0 = 0$, we have $x_8 = 5$ and $y_8 = 0$ again for the first time. Thus, $p = x_{150} = x_6 = -5\sqrt{2}$ and $q = y_{150} = y_6 = 5 + 5\sqrt{2}$. Hence, the final answer is
$5\sqrt{2} + 5(\sqrt{2} + 1) \approx 19.1 \Longrightarrow \boxed{019}$

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Problem 3

Let $ABCD$ and $BCFG$ be two faces of a cube with $AB = 12$. A beam of light emanates from vertex A and reflects off face $BCFG$ at point P, which is 7 units from $\overline{BG}$ and 5 units from $\overline{BC}$. The beam continues to be reflected off the faces of the cube. The length of the light path from the time it leaves point A until it next reaches a vertex of the cube is given by $m\sqrt{n}$, where m and n are integers and n is not divisible by the square of any prime. Find $m + n$.

(A) 200 **(B)** 210 **(C)** 220 **(D)** 230 **(E)** 250

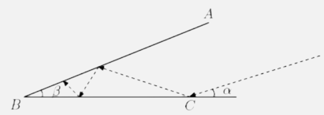
- When a light beam reflects off a surface, the path is like that of a ball bouncing. Picture that, and also imagine X, Y, and Z coordinates for the cube vertices. The coordinates will all involve 0's and 12's only, so that means that the X, Y, and Z distance traveled by the light must all be divisible by 12.
- Since the light's Y changes by 5 and the X changes by 7 (the Z changes by 12, don't worry about that), and 5 and 7 are relatively prime to 12, the light must make 12 reflections onto the XY plane or the face parallel to the XY plane.
- In each reflection, the distance traveled by the light is $\sqrt{(12^2) + (5^2) + (7^2)} = \sqrt{218}$. This happens 12 times, so the total distance is $12\sqrt{218}$. $m = 12$ and $n = 218$, so therefore, the answer is $m + n = \boxed{230}$.

D

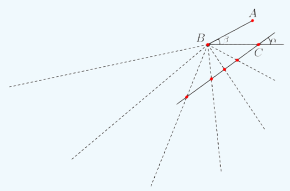
Problem 4

A beam of light strikes $\overline{BC}$ at point C with angle of incidence $\alpha = 19.94°$ and reflects with an equal angle of reflection as shown. The light beam continues its path, reflecting off line segments $\overline{AB}$ and $\overline{BC}$ according to the rule: angle of incidence equals angle of reflection. Given that $\beta = \frac{\alpha}{10} = 1.994°$ and $AB = BC$, determine the number of times the light beam will bounce off the two line segments. Include the first reflection at C in your count.

(A) 71 **(B)** 72 **(C)** 73 **(D)** 74 **(E)** 75



- At each point of reflection, we pretend instead that the light continues to travel straight.



- Note that after k reflections (excluding the first one at C) the extended line will form an angle $k\beta$ at point B. For the kth reflection to be just inside or at point B, we must have $k\beta \leq 180 - 2\alpha \implies k \leq \frac{180-2\alpha}{\beta} = 70.27$. Thus, our answer is, including the first intersection, $\left\lfloor \frac{180-2\alpha}{\beta} \right\rfloor + 1 = \boxed{071}$.

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Problem 5

Lines l_1 and l_2 both pass through the origin and make first-quadrant angles of $\frac{\pi}{70}$ and $\frac{\pi}{54}$ radians, respectively, with the positive x-axis. For any line l, the transformation $R(l)$ produces another line as follows: l is reflected in l_1, and the resulting line is reflected in l_2. Let $R^{(1)}(l) = R(l)$ and $R^{(n)}(l) = R\left(R^{(n-1)}(l)\right)$. Given that l is the line $y = \frac{19}{92}x$, find the smallest positive integer m for which $R^{(m)}(l) = l$.

(A) 940 **(B)** 945 **(C)** 950 **(D)** 955 **(E)** 960

- Let l be a line that makes an angle of θ with the positive x-axis. Let l' be the reflection of l in l_1, and let l'' be the reflection of l' in l_2.
  The angle between l and l_1 is $\theta - \frac{\pi}{70}$, so the angle between l_1 and l' must also be $\theta - \frac{\pi}{70}$. Thus, l' makes an angle of $\frac{\pi}{70} - \left(\theta - \frac{\pi}{70}\right) = \frac{\pi}{35} - \theta$ with the positive x-axis.
- Similarly, since the angle between l' and l_2 is $\left(\frac{\pi}{35} - \theta\right) - \frac{\pi}{54}$, the angle between l'' and the positive x-axis is $\frac{\pi}{54} - \left(\left(\frac{\pi}{35} - \theta\right) - \frac{\pi}{54}\right) = \frac{\pi}{27} - \frac{\pi}{35} + \theta = \frac{8\pi}{945} + \theta$.
- Thus, $R(l)$ makes an $\frac{8\pi}{945} + \theta$ angle with the positive x-axis. So $R^{(n)}(l)$ makes an $\frac{8n\pi}{945} + \theta$ angle with the positive x-axis.
- Therefore, $R^{(m)}(l) = l$ iff $\frac{8m\pi}{945}$ is an integral multiple of π. Thus, $8m \equiv 0 \pmod{945}$. Since $\gcd(8, 945) = 1$, $m \equiv 0 \pmod{945}$, so the smallest positive integer m is $\boxed{945}$.

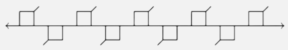
B

Problem 6

The figure below shows line ℓ with a regular, infinite, recurring pattern of squares and line segments. How many of the following four kinds of rigid motion transformations of the plane in which this figure is drawn, other than the identity transformation, will transform this figure into itself?

- some rotation around a point of line ℓ
- some translation in the direction parallel to line ℓ
- the reflection across line ℓ
- some reflection across a line perpendicular to line ℓ

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4



- Statement 1 is true. A 180° rotation about the point half way between an up-facing square and a down-facing square will yield the same figure.
- Statement 2 is also true. A translation to the left or right will place the image onto itself when the figures above and below the line realign (the figure goes on infinitely in both directions).
- Statement 3 is false. A reflection across line ℓ will change the up-facing squares to down-facing squares and vice versa.
- Finally, statement 4 is also false because it will cause the diagonal lines extending from the squares to switch direction. Thus, only 2 statements are true.

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Problem 7

Line l in the coordinate plane has equation $3x - 5y + 40 = 0$. This line is rotated $45°$ counterclockwise about the point $(20, 20)$ to obtain line k. What is the x-coordinate of the x-intercept of line k?

(A) 10 **(B)** 15 **(C)** 20 **(D)** 25 **(E)** 30

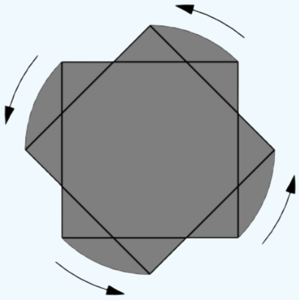
- Since the slope of the line is $\frac{3}{5}$, and the angle we are rotating around is x, then $\tan x = \frac{3}{5}$.
- $\tan(x + 45°) = \frac{\tan x + \tan(45°)}{1 - \tan x \times \tan(45°)} = \frac{0.6+1}{1-0.6} = \frac{1.6}{0.4} = 4$. Hence, the slope of the rotated line is 4.
- Since we know the line intersects the point $(20, 20)$, then we know the line is $y = 4x - 60$. Set $y = 0$ to find the x-intercept, and so $x = \boxed{15}$.

B

Problem 8

A unit square is rotated $45°$ about its center. What is the area of the region swept out by the interior of the square?
(A) $1-\frac{\sqrt{2}}{2}+\frac{\pi}{4}$ **(B)** $\frac{1}{2}+\frac{\pi}{4}$ **(C)** $2-\sqrt{2}+\frac{\pi}{4}$ **(D)** $\frac{\sqrt{2}}{2}+\frac{\pi}{4}$ **(E)** $1+\frac{\sqrt{2}}{4}+\frac{\pi}{8}$

- First, we need to see what this looks like. Below is a diagram.



- For this square with side length 1, the distance from center to vertex is $r=\frac{\sqrt{2}}{2}$, hence the area is composed of a semicircle of radius r, plus 4 times a parallelogram (or a kite with diagonals of $(\sqrt{2}-1)$ and r or $\frac{\sqrt{2}}{2}$) with height $\frac{1}{2}$ and base $\frac{\sqrt{2}}{2(1+\sqrt{2})}$. That is to say, the total area is $\frac{1}{2}\pi\left(\frac{\sqrt{2}}{2}\right)^2+4\frac{\sqrt{2}}{4(1+\sqrt{2})}=2-\sqrt{2}+\frac{\pi}{4}$.

Problem 9

A triangle with vertices $A(0,2)$, $B(-3,2)$, and $C(-3,0)$ is reflected about the x-axis, then the image $\triangle A'B'C'$ is rotated counterclockwise about the origin by $90°$ to produce $\triangle A''B''C''$. Which of the following transformations will return $\triangle A''B''C''$ to $\triangle ABC$?
(A) counterclockwise rotation about the origin by $90°$.
(B) clockwise rotation about the origin by $90°$.
(C) reflection about the x-axis
(D) reflection about the line $y = x$
(E) reflection about the y-axis.

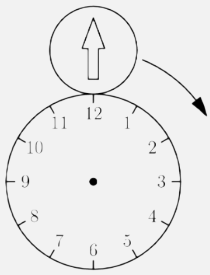
- Consider a point (x, y). Reflecting it about the x-axis will map it to $(x, -y)$.
- Rotating it counterclockwise about the origin by $90°$ will map it to (y, x).
- The operation that undoes this is a reflection about the $y = x$.

D

Problem 10

The diagram below shows the circular face of a clock with radius 20 cm and a circular disk with radius 10 cm externally tangent to the clock face at 12 o' clock. The disk has an arrow painted on it, initially pointing in the upward vertical direction. Let the disk roll clockwise around the clock face. At what point on the clock face will the disk be tangent when the arrow is next pointing in the upward vertical direction?

(A) 2 o' clock **(B)** 3 o' clock **(C)** 4 o' clock **(D)** 6 o' clock **(E)** 8 o' clock



- he rotation factor of the arrow is the sum of the rates of the regular rotation of the arrow (every 360° rotation = 1).
- The rotation of the disk around the clock with twice the circumference (every 360° rotation = 2).
- Thus, the rotation factor of the arrow is 3, and so our answer corresponds to
$$\frac{360^\circ}{3} = 120^\circ$$
, which is 4 o' clock.

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