1983 AIME

1. Let x, y, and z all exceed 1, and let w be a positive number such that
$$\log_x w = 24,\ \log_y = 40,\ \text{and}\ \log_{xyz} w = 12.$$
Find $\log_z w$.

2. Let $f(x) = |x - p| + |x - 15| + |x - p - 15|$, where $0 < p < 15$. Determine the minimum value taken by $f(x)$ for x in the interval $p \leq x \leq 15$.

3. What is the product of the real roots of the equation
$$x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}\ ?$$

4. A machine-shop cutting tool has the shape of a notched circle, as shown. The radius of the circle is $\sqrt{50}$ cm, the length of AB is 6 cm and that of BC is 2 cm. The angle ABC is a right angle. Find the square of the distance (in centimeters) from B to the center of the circle.

5. Suppose that the sum of the squares of two complex numbers x and y is 7 and the sum of the cubes is 10. What is the largest real value that $x + y$ can have?

6. Let $a_n = 6^n + 8^n$. Determine the remainder on dividing a_{83} by 49.

7. Twenty five of King Arthur's knights are seated at their customary round table. Three of them are chosen – all choices of three being equally likely – and are sent off to slay a troublesome dragon. Let P be the probability that at least two of the three had been sitting next to each other. If P is written as a fraction in lowest terms, what is the sum of the numerator and denominator?

8. What is the largest 2-digit prime factor of the integer $n = \binom{200}{100}$?

9. Find the minimum value of
$$f(x) = \frac{9x^2 \sin^2 x + 4}{x \sin x}$$
for $0 < x < \pi$.

10. The numbers 1447, 1005, and 1231 have something in common: each is a 4-digit number beginning with 1 that has exactly two identical digits. How many such numbers are there?

11. The solid shown has a square base of side length s. The upper edge is parallel to the base and has length $2s$. All other edges have length s. Given that $s = 6\sqrt{2}$, what is the volume of the solid? [The solid has six faces. It has a square base $ABCD$, two triangular sides BCE and ADF, and two trapezoidal sides $ABEF$ and $CDFE$. The upper edge is EF.]

12. Diameter AB of a circle has length a 2-digit integer (base ten). Reversing the digits gives the length of a perpendicular chord CD. The distance from their intersection point H to the center O is a positive rational number. Determine the length of AB.

13. For $\{1, 2, 3, \ldots, n\}$ and each of its nonempty subsets a unique **alternating sum** is defined as follows: Arrange the number in the subset in decreasing order and then, beginning with the largest, alternately add and subtract successive numbers. (For example, the alternating sum for $\{1, 2, 3, 6, 9\}$ is $9 - 6 + 4 - 2 + 1 = 6$ and for $\{5\}$ it is simply 5. Find the sum of all such alternating sums for $n = 7$.

14. In the adjoining figure, two circles of radii 8 and 6 are drawn with their centers 12 units apart. At P, one of the points of intersection, a line is drawn in such a way that the chords QP and PR have equal length. [P is the midpoint of QR.] Find the square of the length of QP.

15. The adjoining figure shows two intersecting chords in a circle, with B on minor arc AD. Suppose that the radius of the circle is 5, that $BC = 6$, and that AD is bisected by BC. Suppose further that AD is the only chord starting at A which is bisected by BC. It follows that the sine of the central angle of minor arc AB is a rational number. If this number is expressed as a fraction m/n in lowest terms, what is the product mn?

1984 AIME

1. Find the value of $a_2 + a_4 + a_6 + \cdots + a_{98}$ if $a_1, a_2, a_3, \ldots$ is an arithmetic progression with common difference 1, and $a_1 + a_2 + a_3 + \cdots + a_{98} = 137$.

2. The integer n is the smallest positive multiple of 15 such that every digit of n is either 0 or 8. Compute $\frac{n}{15}$.

3. A point P is chosen in the interior of ΔABC so that when lines are drawn through P parallel to the sides of ΔABC, the resulting smaller triangles, t_1, t_2, and t_3 in the figure, have areas 4, 9, and 49, respectively. Find the area of ΔABC.

4. Let S be a list of positive integers – not necessarily distinct – in which the number 68 appears. The average (arithmetic mean) of the numbers in S is 56. However, if 68 is removed, the average of the remaining numbers drops to 55. What is the largest number that can appear in S?

5. Determine the value of ab if

$$\log_8 a + \log_4 b^2 = 5 \quad \text{and} \quad \log_8 b + \log_4 a^2 = 7.$$

6. Three circles, each of radius 3, are drawn with centers at $(14, 92)$, $(17, 76)$, and $(19, 84)$. A line passing through $(17, 76)$ is such that the total area of the parts of the three circles to one side of the line is equal to the total area of the parts of the three circles to the other side of it. What is the absolute value of the slope of this line?

7. The function f is defined on the set of integers and satisfies

$$f(n) = \begin{cases} n - 3, & \text{if } n \geq 1000. \\ f(f(n+5)), & \text{if } n < 1000. \end{cases}$$

Find $f(84)$.

8. The equation $z^6 + z^3 + 1$ has complex root with argument (angle) θ between 90° and 180° in the complex plane. Determine the degree measure of θ.

9. In tetrahedron $ABCD$, edge AB has length 3 cm. The area of face ABC is 15 cm^2 and the area of face ABD is 12 cm^2. These two faces meet each other at a 30° angle. Find the volume of the tetrahedron in cm^3.

10. Mary told John her score on the American High School Mathematics Examination (AHSME), which was over 80. From this, John was able to determine the number of problems Mary solved correctly. If Mary's score had been any lower, but still over 80, John could not have determined this. What was Mary's score? (Recall that the AHSME consists of 30 multiple-choice problems and that one's score, s, is computed by the formula $s = 30 + 4c - w$, where c is the number of correct and w is the number of wrong answers; students are not penalized for problems left unanswered.)

11. A gardener plants three maple trees, four oak trees and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Let $\frac{m}{n}$ in lowest terms be the probability that no two birch trees are next to one another. Find $m + n$.

12. A function f is defined for all real numbers and satisfies

$$f(2 + x) = f(2 - x) \qquad \text{and} \qquad f(7 + x) = f(7 - x)$$

for all x. If $x = 0$ is a root of $f(x) = 0$, what is the least number of roots $f(x) = 0$ must have in the interval $-1000 \leq x \leq 1000$?

13. Find the value of $10 \cot(\cot^{-1} 3 + \cot^{-1} 7 + \cot^{-1} 13 + \cot^{-1} 21)$.

14. What is the largest even integer which cannot be written as the sum of two odd composite numbers? (Recall that a positive integer is said to be composite if it is divisible by at least one positive integer other than 1 and itself.)

15. Determine $x^2 + y^2 + z^2 + w^2$ if

$$\begin{aligned}
\frac{x^2}{2^2-1} + \frac{y^2}{2^2-3^2} + \frac{z^2}{2^2-5^2} + \frac{w^2}{2^2-7^2} &= 1, \\
\frac{x^2}{4^2-1} + \frac{y^2}{4^2-3^2} + \frac{z^2}{4^2-5^2} + \frac{w^2}{4^2-7^2} &= 1, \\
\frac{x^2}{6^2-1} + \frac{y^2}{6^2-3^2} + \frac{z^2}{6^2-5^2} + \frac{w^2}{6^2-7^2} &= 1, \\
\frac{x^2}{8^2-1} + \frac{y^2}{8^2-3^2} + \frac{z^2}{8^2-5^2} + \frac{w^2}{8^2-7^2} &= 1.
\end{aligned}$$

1985 AIME

1. Let $x_1 = 97$, and for $n > 1$ let $x_n = \dfrac{n}{x_{n-1}}$. Calculate the product $x_1 x_2 \ldots x_8$.

2. When a right triangle is rotated about one leg, the volume of the cone produced is 800π cm^3. When the triangle is rotation about the other leg, the volume of the cone produced is 1920π cm^3. What is the length (in cm) of the hypotenuse of the triangle?

3. Find c if a, b, and c are positive integers which satisfy $(a + bi)^3 - 107i$, where $i^2 = -1$.

4. A small square is constructed inside a square of area 1 by dividing each side of the unit square into n equal parts, and then connecting the vertices to the division points closest to the opposite vertices, as shown in the figure on the right. Find the value of n if the the area of the small square (shaded in the figure) is exactly 1/1985. (The vertices are $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$, and $D = (0, 1)$. We select points $A' = (1/n, 0)$, $B' = (1, 1/n)$, $C' = (1 - 1/n, 1)$, and $D' = (0, 1 - 1/n)$ on AB, BC, CD, and DA respectively. The small square is bounded by the lines AC', BD', CA', and DB'.)

5. A sequence of integers $a_1, a_2, a_3, \ldots$ is chosen so that $a_n = a_{n-1} - a_{n-2}$ for each $n \geq 3$. What is the sum of the first 2001 terms of this sequence if the sum of the first 1492 terms is 1985, and the sum of the first 1985 terms is 1492?

6. As shown in the figure on the right, ΔABC is divided into six smaller triangles by lines [AA', BB', and CC'] drawn from the vertices through a common interior point [P]. The areas of four of these triangles are as indicated. [$AA'P = 40$, $A'BP = 30$, $BB'P = 35$, and $CC'P = 84$.] Find the area of ΔABC.

7. Assume that a, b, c, and d are positive integers such that $a^5 = b^4$, $c^3 = d^2$, and $c - a = 19$. Determine $d - b$.

8. The sum of the following seven numbers is exactly 19:

$$a_1 = 2.56, \qquad a_2 = 2.61, \qquad a_3 = 2.65, \qquad a_4 = 2.71,$$

$$a_5 = 2.79, \qquad a_6 = 2.82, \qquad a_7 = 2.86.$$

It is desired to replace each a_i by an integer approximation A_i, $1 \leq i \leq 7$, so that the sum of the A_i's is also 19, and so that M, the maximum of the "errors" $|A_i - a_i|$, is as small as possible. For this minimum M, what is $100M$?

9. In a circle, parallel chords of lengths 2, 3, and 4 determine central angles of α, β, and $\alpha + \beta$ radians, respectively, where $\alpha + \beta < \pi$. If $\cos \alpha$, which is a positive rational number, is expressed as a fraction in lowest terms, what is the sum of its numerator and denominator?

10. How many of the first 1000 positive integers can be expressed in the form

$$\lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 6x \rfloor + \lfloor 8x \rfloor,$$

where x is a real number, and $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z?

11. An ellipse has foci at $(9, 20)$ and $(49, 55)$ in the xy-plane and is tangent to the x-axis. What is the length of its major axis?

12. Let A, B, C and D be the vertices of a regular tetrahedron, each of whose edges measures 1 meter. A bug, starting from vertex A, observes the following rule: at each vertex if chooses one of the three edges meeting at that vertex, and crawls along that edge to the vertex at its opposite end. Let $p = n/729$ be the probability that the bug is at vertex A when it has crawled exactly 7 meters. Find the value of n.

13. The numbers in the sequence $101, 104, 109, 116, \ldots$ are of the form $a_n = 100 + n^2$, where $n = 1, 2, 3, \ldots$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

14. In a tournament each player played exactly one game against each of the other players. In each game the winner was awarded 1 point, the loser got 0 points, and each of the two players earned 1/2 point if the game was a tie. After the completion of the tournament, it was found that exactly half of the points earned by each player were earned in games against the ten players with the least number of points. (In particular, each of the ten lowest scoring players earned half of her/his points against the other nine of the ten). What was the total number of players in the tournament?

15. Three 12cm $\times$ 12cm squares are each cut into two pieces A and B, as shown in the first figure below, by joining the midpoints of two adjacent sides. These six pieces are then attached to a regular hexagon, as shown in the second figure, so as to fold into a polyhedron. What is the volume (in cm^3) of this polyhedron?

1986 AIME

1. What is the sum of the solutions of the equation

$$\sqrt[4]{x} = \frac{12}{7 - \sqrt[4]{x}} \ ?$$

2. Evaluate the product

$$(\sqrt{5} + \sqrt{6} + \sqrt{7})(\sqrt{5} + \sqrt{6} - \sqrt{7})(\sqrt{5} - \sqrt{6} + \sqrt{7})(-\sqrt{5} + \sqrt{6} + \sqrt{7}).$$

3. If $\tan x + \tan y = 25$ and $\cot x + \cot y = 30$, what is $\tan(x + y)$?

4. Determine $3x_4 + 2x_5$, if x_1, x_2, x_3, x_4 and x_5 satisfy the system of equations given below:

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 + x_5 &= 6 \\ x_1 + 2x_2 + x_3 + x_4 + x_5 &= 12 \\ x_1 + x_2 + 2x_3 + x_4 + x_5 &= 24 \\ x_1 + x_2 + x_3 + 2x_4 + x_5 &= 48 \\ x_1 + x_2 + x_3 + x_4 + 2x_5 &= 96 \end{aligned}$$

5. What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

6. The pages of a book are numbered 1 through n. When the page numbers of the book were added, one of the page numbers was mistakenly added twice, resulting in the incorrect sum of 1986. What was the number of the page that was added twice?

7. The increasing sequence $1, 3, 4, 9, 10, 12, 13, \ldots$ consists of all those positive integers which are powers of 3 or sums of distinct powers of 3. Find the 100^{th} term of this sequence (where 1 is the 1^{st} term, 3 is the 2^{nd} term, and so on).

8. Let S be the sum of the base 10 logarithms of all of the proper divisors of 1,000,000. By a proper divisor of a natural number we mean a positive integer divisor other than 1 and the number itself. What is the integer nearest to S?

9. In ΔABC shown below, $AB = 425$, $BC = 450$, and $CA = 510$. Moreover, P is an interior point chosen so that the segments DE, FG, and HI are each of length d, contain P, and are parallel to the sides AB, BC, and CA, respectively. Find d.

10. In a parlor game the "magician" asks one of the participants to think of a three digits number (abc) where a, b and c represent digits in base 10 in the order indicated. Then the magician asks this person to form the numbers $(acb), (bac), (bca), (cab)$ and (cba), to add these five numbers, and to reveal their sum, N. If told the value of N, the magician can identify the original number, (abc). Play the role of the magician and determine (abc) if $N = 3194$.

11. The polynomial $1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$ may be written in the form $a_0 + a_1 y + a_2 y^2 + a_3 y^3 + \cdots + a_{16} y^{16} + a_{17} y^{17}$, where $y = x + 1$ and the a_i's are constants. Find the value of a_2.

12. Let the sum of a set of numbers be the sum of its elements. Let S be a set of positive integers, none greater than 15. Suppose no two disjoint subsets of S have the same sum. What is the largest sum a set S with these properties can have?

13. In a sequence of coin tosses one can keep a record of the number of instances when a tail is immediately followed by a head, a head is immediately followed by a head, etc. We denote these by TH, HH, etc. For example, in the sequence HHTTHHHHTHHTTTT of 15 coin tosses we observe that there are 5 HH, three HT, two TH and four TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH and five TT subsequences?

14. The shortest distances between an interior diagonal of a rectangular parallelepiped (box), P, and the edges it does not meet are $2\sqrt{5}$, $30/\sqrt{13}$, and $15/\sqrt{10}$. Determine the volume of P.

15. Let ΔABC be a right triangle in the xy-plane with the right angle at C. Given that the length of the hypotenuse AB is 60, and that the medians through A and B lie along the lines $y = x + 3$ and $y = 2x + 4$, respectively, find the area of ΔABC.

1987 AIME

1. An ordered pair (m, n) of non-negative integers is called "simple" if the addition $m + n$ in base 10 requires no carrying. Find the number of simple ordered pairs of non-negative integers that sum to 1492.

2. What is the largest possible distance between two points, one on the sphere of radius 19 with center $(-2, -10, 5)$ and the other on the sphere of radius 87 with center $(12, 8, -16)$?

3. By a proper divisor of a natural number we mean a positive integral divisor other than 1 and the number itself. A natural number greater than 1 will be called "nice" if it is equal to the product of its distinct proper divisors. What is the sum of the first ten nice numbers?

4. Find the area of the region enclosed by the graph of $|x - 60| + |y| = |x/4|$.

5. Find $3x^2y^2$ if x and y are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$.

6. Rectangle $ABCD$ is divided into four parts of equal area by five segments as shown in the figure, where $XY = YB + BC + CZ = ZW = WD + DA + AX$, and PQ is parallel to AB. Find the length of AB (in cm) if $BC = 19$ cm and $PQ = 87$ cm. [X lies on AB, Y lies on XB, W lies on DC, and Z lies on WC.]

7. Let $[r, s]$ denote the least common multiple of positive integers r and s. Find the number of ordered triples (a, b, c) for which $[a, b] = 1000, [b, c] = 2000$, and $[c, a] = 2000$.

8. What is the largest positive integer n for which there is a unique integer k such that $\frac{8}{15} < \frac{n}{n+k} < \frac{7}{13}$?

9. Triangle ABC has right angle at B, and contains a point P for which $PA = 10$, $PB = 6$, and $\angle APB = \angle BPC = \angle CPA$. Find PC.

10. Al walks to down to the bottom of an escalator that is moving up and he counts 150 steps. His friend, Bob, walks up to the top of the escalator and counts 75 steps. If Al's speed of walking (in steps per unit time) is three times Bob's speed, how many steps are visible on the escalator at any given time? (Assume that this number is constant.)

11. Find the largest possible value of k for which 3^{11} is expressible as the sum of k consecutive integers.

12. Let m be the smallest positive integer whose cube root is of the form $n+r$, where n is a positive integer and r is a positive real number less than $1/1000$. Find n.

13. A given sequence $r_1, r_2, \ldots, r_n$ of distinct real numbers can be put in ascending order by means of one or more "bubble passes". A bubble pass through a given sequence consists of comparing the second term with the first term and exchanging them if and only if the second term is smaller, then comparing the third term with the current second term and exchanging them if and only if the third term is smaller, and so on in order, through comparing the last term, r_n, with its current predecessor and exchanging them if and only if the last term is smaller.

    Suppose that $n = 40$, and that the terms of the initial sequence $r_1, r_2, \ldots, r_{40}$ are distinct from one another and are in random order. Let p/q, in lowest terms, be the probability that the number that begins as $r_2 0$ will end up, after one pass, in the 30^{th} place (i.e., will have 29 terms on its left and 10 terms on its right). Find $p + q$.

14. Compute
$$\frac{(10^4 + 324)(22^4 + 324)(34^4 + 324)(46^4 + 324)(58^4 + 324)}{(4^4 + 324)(16^4 + 324)(28^4 + 324)(40^4 + 324)(52^4 + 324)}$$

15. Squares S_1 and S_2 are inscribed in right triangle ABC, as shown in the figures below. Find $AC + BC$ if area$(S_1) = 441$ and area$(S_2) = 440$. [The right angle is at C. S_1 is in the corner, with one vertex at C and the opposite vertex on AB. S_2 has one side along AB, and the other two vertices of S_2 lie on AC and BC respectively.]

1988 AIME

1. One commercially available ten button lock may be opened by depressing – in any order – the correct five buttons. The sample shown at right has $\{1, 2, 3, 6, 9\}$ as its combination. Suppose that these locks are redesigned so that sets of as many as nine buttons or as few as one button could serve as combinations. How many additional combinations would this allow?

2. For any positive integer k, let $f_1(k)$ denote the square of the sum of the digits of k. For $n \geq 2$, let $f_n(k) = f_1(f_{n-1}(k))$. Find $f_{1998}(11)$.

3. Find $(\log_2 x)^2$ if $\log_2(\log_8 x) = \log_8(\log_2 x)$.

4. Suppose that $|x_1| < 1$ for $i = 1, 2, \ldots, n$. Suppose further that
$$|x_1| + |x_2| + \ldots + |x_n| = 19 + |x_1 + x_2 + \ldots + x_n|.$$
    What is the smallest possible value of n?

5. Let m/n, in lowest terms, be the probability that a randomly chosen positive divisor of 10^{99} is an integer multiple of 10^{88}. Find $m + n$.

6. It is possible to place positive integers into the vacant twenty-one squares of the 5×5 square shown on the right so that the numbers in each row and column form arithmetic sequences. Find the number that must occupy the vacant square marked by the asterisk (*). [In the figure, $A(2,2) = 74$, $A(3,5) = 186$, $A(4,3) = 103$, and $A(5,1) = 0$, where $A(i,j)$ denotes the entry in the i-th row and the j-th column. The vacant square is $A(1,4)$.]

7. In ΔABC, $\tan(\angle CAB) = 22/7$ and the altitude from A divides BC into segments of lengths 3 and 17. What is the area of ΔABC?

8. The function f, defined on the set of ordered pairs of positive integers, satisfies the following properties:

$$f(x,x) = x, f(x,y) = f(y,x), \text{ and } (x+y)f(x,y) = yf(x,x+y).$$

Calculate $f(14,54)$.

9. Find the smallest positive integer whose cube ends in 888.

10. A convex polyhedron has for its faces 12 squares, 8 regular hexagons, and 6 regular octagons. At each vertex of the polyhedron one square, one hexagon, and one octagon meet. How many segments joining vertices of the polyhedron lie in the interior of the polyhedron rather than along an edge or a face?

11. Let $w_1, w_2, \ldots w_n$ be complex numbers. A line L in the complex plane is called a *mean line* for the points $w_1, w_2, \ldots, w_n$ if L contains points (complex numbers) $z_1, z_2, \ldots, z_n$ such that

$$\sum_{k=1}^{n}(z_k - w_k) = 0.$$

For the numbers $w_1 = 32 + 170i$, $w_2 = -7 + 64i$, $w_3 = -9 + 200i$, $w_4 = 1 + 27i$, and $w_5 = -14 + 43i$ there is a unique mean line with y-intercept 3. Find the slope of the mean line.

12. Let P be an interior point ΔABC and extend lines from the vertices through P to the opposite sides. Let a, b, c, and d denote the lengths of the segments indicated in the figure. Find the product abc if $a + b + c = 43$ and $d = 3$. [$a = AP$, $b = BP$, $c = CP$, and $d = A'P = B'P = C'P$; where A', B', and C' are the points on the triangle such that AA', BB', and CC' all meet at P.]

13. Find a if a and b are integers such that $x^2 - x - 1$ is a factor of $ax^{17} + bx^{16} + 1$.

14. Let C be the graph of $xy = 1$, and denote by C^* the reflection of C in the line $y = 2x$. Let the equation of C^* be written in the form

$$12x^2 + bxy + cy^2 + d = 0.$$

Find the product bc.

15. In an office at various times during the day, the boss gives the secretary a letter to type, each time putting the letter on top of the pile in the secretary's in-box. When there is time, the secretary takes the top letter off the pile and types it. There are nine letters to be typed during the day, and the boss delivers them in the order $1, 2, 3, 4, 5, 6, 7, 8, 9$.

While leaving for lunch, the secretary tells a colleague that letter 8 has already been typed, but says nothing else about the morning's typing. The colleague wonders which of the nine letters remain to be typed after lunch and in what order they will be typed. Based upon the above information, how many such *after-lunch typing orders* are possible? (That there are no letters left to be typed is one of the possibilities.)

1991 AIME

1. Find x^2+y^2 if x and y are positive integers such that

$$\begin{aligned} xy+x+y &= 71 \\ x^2y+xy^2 &= 880. \end{aligned}$$

2. Rectangle $ABCD$ has sides $\overline{AB}$ of length 4 and $\overline{CB}$ of length 3. Divide $\overline{AB}$ into 168 congruent segments with points $A=P_0, P_1, \ldots, P_{168}=B$, and divide $\overline{CB}$ into 168 congruent segments with points $C=Q_0, Q_1, \ldots, Q_{168}=B$. For $1 \le k \le 167$, draw the segments $\overline{P_K Q_K}$. Repeat this construction on the lines $\overline{AD}$ and $\overline{CD}$, and then draw the diagonal $\overline{AC}$. Find the sum of the lengths of the 335 parallel segments drawn.

3. Expanding $(1+0.2)^{1000}$ by the binomial theorem gives

$$\binom{1000}{0}(0.2)^0+\binom{1000}{1}(0.2)^1+\binom{1000}{2}(0.2)^2+\cdots+\binom{1000}{1000}(0.2)^{1000}=A_0+A_1+A_2+\cdots+A_{1000},$$

where $A_k=\binom{1000}{k}(0.2)^k$. For which k is A_k the largest?

4. How many real numbers x satisfy the equation

$$\frac{1}{5}\log_2 x=\sin(5\pi x)?$$

5. Given a rational number, write it as a fraction in lowest terms and calculate the product of the resulting numerator and denominator. For how many rational numbers between 0 and 1 will 20! be the resulting product?

6. Suppose r is a real number for which

$$\left\lfloor r+\frac{19}{100}\right\rfloor+\left\lfloor r+\frac{20}{100}\right\rfloor+\left\lfloor r+\frac{21}{100}\right\rfloor+\cdots+\left\lfloor r+\frac{91}{100}\right\rfloor=546.$$

Find $\lfloor 100r\rfloor$. (For real x, $\lfloor x\rfloor$ is the greatest integer less than or equal to x.)

7. Find A^2, where A is the sum of the absolute values of all solutions of the following equation:

$$x=\sqrt{19}+\frac{91}{\sqrt{19}+\frac{91}{\sqrt{19}+\frac{91}{\sqrt{19}+\frac{91}{\sqrt{19}+\frac{91}{x}}}}}$$

8. For how many real numbers a does the equation $x^2+ax+6a=0$ have only integer solutions for x?

9. Suppose that $\sec x+\tan x=\frac{22}{7}$ and that $\csc x+\cot x=\frac{m}{n}$, where $\frac{m}{n}$ is in lowest terms. Find $m+n$.

10. Two three letter strings, *aaa* and *bbb*, are transmitted electronically. Each string is sent letter by letter. Due to faulty equipment, each of the six letters has a 1/3 chance of being received incorrectly as the other letter. Whether a given letter is received correctly or incorrectly is independent of the reception of any other letter. Let S_a be the three letter string received when *aaa* is transmitted, and let S_b be the three letter string received when *bbb* is transmitted. Let p be the probability that S_a comes before S_b in alphabetical order. When p is written as a fraction in lowest terms, what is its numerator?

11. Twelve congruent disks are placed on a circle C of radius 1 in such a way that the twelve disks cover the circle C, no two of the disks overlap, and each of the twelve disks is tangent to its two neighbors. The sum of the areas of the twelve disks can be written in the form $\pi(a-b\sqrt{c})$ where a,b,c are positive integers and c is not divisible by the square of any prime. Find $a+b+c$.

12. Rhombus $PQRS$ is inscribed in rectangle $ABCD$ so that the vertices P, Q, R, and S are interior points on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ respectively. It is given that $PB = 15$, $BQ = 20$, $PR = 30$, and $QS = 40$. Let m/n, in lowest terms, denote the perimeter of $ABCD$. Find $m + n$.

13. A drawer contains a mixture of red socks and blue socks, at most 1991 in all. It happens that, when two socks are selected randomly without replacement, there is a probability of exactly 1/2 that both are red or both are blue. What is the largest possible number of red socks in the drawer?

14. A hexagon is inscribed in a circle. Five of the sides have length 81, and the sixth side $\overline{AB}$ has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A.

15. For any positive integer n, define S_n to be the minimum value of the sum

$$\sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2},$$

where $a_1, a_2, \ldots, a_n$ are positive real numbers that sum to 17. There is a unique positive integer N for which S_N is also an integer. Find N.

1997 AIME

1. How many of the integers between 1 and 1000, inclusive, can be expressed as the difference of the squares of two nonnegative integers?

2. The nine horizontal and nine vertical lines on an 8×8 checkerboard form r rectangles, of which s are squares. The number s/r can be written in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

3. Sarah intended to multiply a two-digit number and a three-digit number, but she left out the multiplication sign and simply placed the two-digit number to the left of the three-digit number, thereby forming a five-digit number. This number is exactly nine times the product Sarah should have obtained. What is the sum of the two-digit number and the three-digit number?

4. Circles of radii 5, 5, 8, and m/n are mutually externally tangent, where m and n are relatively prime positive integers. Find $m + n$.

5. The number r can be expressed as a four-place decimal $0.abcd$, where a, b, c, and d represent digits, any of which could be zero. It is desired to approximate r by a fraction whose numerator is 1 or 2 and whose denominator is an integer. The closest such fraction to r is $2/7$. What is the number of possible values for r?

6. Point B is in the exterior of the regular n-sided polygon $A_1A_2 \ldots A_n$, and A_1A_2B is an equilateral triangle. What is the largest value of n for which A_n, A_1, and B are consecutive vertices of a regular polygon?

7. A car travels due east at 2/3 mile per minute on a long, straight road. At the same time, a circular storm, whose radius is 51 miles, moves southeast at $\frac{\sqrt{2}}{2}$ miles per minute. At time $t = 0$, the center of the storm is 110 miles due north of the car. At time $t = t_1$ minutes, the car enters the storm circle, and at time $t = t_2$ minutes, the car leaves the storm circle. Find $(t_1 + t_2)/2$.

8. How many different 4×4 arrays whose entries are all 1's and -1's have the property that the sum of the entries in each row is 0 and the sum of the entries in each column is 0?

9. Given a nonnegative real number x, let $\langle x \rangle$ denote the fractional part of x; that is, $\langle x \rangle = x - [x]$, where $[x]$ denotes the greatest integer less than or equal to x. Suppose that a is positive, $\langle a^{-1} \rangle = \langle a^2 \rangle$, and $2 < a^2 < 3$. Find the value of $a^{12} - 144a^{-1}$.

10. Every card in a deck has a picture of one shape − circle, square, or triangle, which is painted in one of three colors − red, blue, or green. Furthermore, each color is applied in one of three shades − light, medium, or dark. The deck has 27 cards, with every shape-color-shade combination represented. A set of three cards from the deck is called complementary if all of the following statements are true:

    (a) Either each of the three cards has a different shape or all three of the cards have the same shape.
    (b) Either each of the three cards has a different color or all three of the cards have the same color.
    (c) Either each of the three cards has a different shade or all three of the cards have the same shade.

    How many different complementary three-card sets are there?

11. Let $x = \dfrac{\sum_{n=1}^{44} \cos n}{\sum_{n=1}^{44} \sin n}$. What is the greatest integer that does not exceed $100x$?

12. The function f is defined by $f(x) = \dfrac{ax+b}{cx+d}$ where a, b, c, and d are nonzero real numbers, has the properties $f(19) = 19$, $f(97) = 97$, and $f(f(x)) = x$ for all values of x except $-d/c$. Find the unique number that is not in the range of f.

13. Let S be the set of points in the plane that satisfy

$$\Big|\big||x|-2\big|-1\Big|+\Big|\big||y|-2\big|-1\Big|=1.$$

If a model of S were built from wire of negligible thickness, then the total length of wire required would be $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime number. Find $a+b$.

14. Let v and w be distinct, randomly chosen roots of the equation $z^{1997}-1=0$. Let m/n be the probability that $\sqrt{2+\sqrt{3}} \le |v+w|$, where m and n are relatively prime positive integers. Find $m+n$.

15. The sides of rectangle $ABCD$ have lengths 10 and 11. An equilateral triangle is drawn so that no point of the triangle lies outside $ABCD$. The maximum possible area of such a triangle can be written in the form $p\sqrt{q}-r$, where p, q, and r are positive integers, and q is not divisible by the square of any prime number. Find $p+q+r$.

AIME Answer Key

1983 AIME

1.	060	6.	035	11.	288
2.	015	7.	057	12.	065
3.	020	8.	061	13.	448
4.	026	9.	012	14.	130
5.	004	10.	432	15.	175

1984 AIME

1.	093	6.	024	11.	106
2.	592	7.	997	12.	401
3.	144	8.	160	13.	015
4.	649	9.	020	14.	038
5.	512	10.	119	15.	036

1985 AIME

1.	384	6.	315	11.	085
2.	026	7.	757	12.	182
3.	198	8.	061	13.	401
4.	032	9.	049	14.	025
5.	986	10.	600	15.	864

1986 AIME

1.	337	6.	033	11.	816
2.	104	7.	981	12.	061
3.	150	8.	141	13.	560
4.	181	9.	306	14.	750
5.	890	10.	358	15.	400

1987 AIME

1.	300	6.	193	11.	486
2.	137	7.	070	12.	019
3.	182	8.	112	13.	931
4.	480	9.	033	14.	373
5.	588	10.	120	15.	462

1988 AIME

1.	770	6.	142	11.	163
2.	169	7.	110	12.	441
3.	027	8.	364	13.	987
4.	020	9.	192	14.	084
5.	634	10.	840	15.	704

1997 AIME

1.	750	6.	042	11.	241
2.	125	7.	198	12.	058
3.	126	8.	090	13.	066
4.	017	9.	233	14.	582
5.	417	10.	117	15.	554